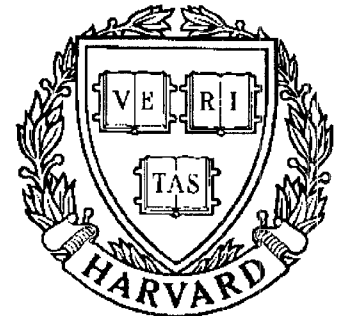


# TECHNICAL RESEARCH REPORT



S Y S T E M S  
R E S E A R C H  
C E N T E R



*Supported by the  
National Science Foundation  
Engineering Research Center  
Program (NSFD CD 8803012),  
the University of Maryland,  
Harvard University,  
and Industry*

## **A Multibody Analog of Dual-Spin Problems**

*by L.-S. Wang and P.S. Krishnaprasad*

Report Documentation Page				Form Approved OMB No. 0704-0188	
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE <b>1990</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-1990 to 00-00-1990</b>	
4. TITLE AND SUBTITLE <b>A Multibody Analog of Dual-Spin Problems</b>				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>University of Maryland, Systems Research Center, College Park, MD, 20742</b>				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT <b>see report</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES <b>20</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

# A MULTIBODY ANALOG OF DUAL-SPIN PROBLEMS \*

Li-Sheng Wang

P.S. Krishnaprasad

Electrical Engineering Department

&

Systems Research Center

University of Maryland, College Park.

ABSTRACT. In the field of spacecraft attitude control and stabilization, the dual-spin maneuver has an important place. Here, we consider a problem which is a multibody analog of the dual-spin problem. The dynamical equations are derived using a modified form of the Euler-Lagrange equation on the special orthogonal group  $SO(3)$ . It is then shown that with a suitable damping mechanism on one body and on the joint, an asymptotic stability theorem can be concluded by using the LaSalle invariance principle .

---

\* This work was supported in part by the AFOSR University Research Initiative Program under grants AFOSR- 87-0073 and AFOSR-90-0105 and by the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012, and also by the Army Research Office through the Mathematical Sciences Institute at Cornell University.



## 1 Introduction

In designing a communication satellite or an interplanetary probe, engineers are often faced with the requirement that the spacecraft be able to maintain a fixed orientation relative to some inertial frame. The dual-spin technique is a simple, commonly used technique for meeting this requirement. A dual-spin spacecraft consists of the spacecraft body and on-board motor-driven symmetric rotors. In the presence of a suitable damping mechanism and for sufficiently high rotor velocities, the attitude acquisition can be achieved. The final state is a steady spin about a fixed axis. This single rigid body dual-spin problem has been studied extensively before, e.g. [8] [7] [5] and the references therein. A rigorous proof of asymptotic atability can be found in [8].

In this paper, we study a multibody analog of dual-spin problems. We consider a system (see Fig. 1) consisting of two rigid bodies connected by a ball-in-socket joint. There are also three rigid symmetric rotors mounted on the centers of mass of each body. One set of the rotors, called *driven rotors*, are set in a constant relative motion to the carrier body. The other set of rotors carried by the second body provide damping torques to the overall motion. These are called *damping rotors*. We shall prove that with an additional damping mechanism on the ball-in-socket joint, the motion is asymptotically stable and approaches the stable equilibria of the coupled system. In deriving the dynamical equations, we will use explicit representations of higher order tangent bundles of the special orthogonal group  $SO(3)$ .

In the following, we shall describe the system configuration and compute the Lagrangian of the system in Section 2. We then discuss a version of the Euler-Lagrange equation for the motion on the special orthogonal group  $SO(3)$  in Section 3. We will generalize this approach to derive the dynamical equations for our problem. In Section 5 we shall prove an asymptotic stability theorem by using a Lyapunov function and invoking the LaSalle invariance principle.

The problems of multibody systems in space are very complicated. The dynamical

behaviors of these systems are highly coupled, highly nonlinear, and poorly understood. The works of [13] [10] [6] help us to understand the phase portraits of problems such as the one discussed in this paper. Also, the underlying control problems associated to multibody systems are quite complex. Controllability questions for rigid bodies with and without rotors were addressed in [5] [3] [4] [2]. Similar results for multibody problems are desirable. Here we simply set up the Lagrangian framework, formulate a basic control problem and then prove an asymptotic stability theorem for a multibody system under a dual-spin feedback law. The questions about attitude acquisition and rates of convergence are under investigation and we hope to report on these in a later paper.

## 2. System Configuration

We consider a system of two rigid bodies connected by a three-degree-of-freedom spherical joint with rotors mounted on each body. The system under consideration is shown in Fig. 1. The inertial observer is at the center of mass of the system of bodies.

Ignoring the specific kinematic relationships between the rotors and the bodies, the unconstrained configuration space  $Q_u$  is parametrized by the attitudes of these eight bodies,

$$\begin{aligned} Q_u &= \{ (B_1, S_1, S_2, S_3, B_2, D_1, D_2, D_3) \}, \\ &= SO(3) \times (SO(3))^3 \times SO(3) \times (SO(3))^3. \end{aligned}$$

To account for the body-rotor relations, we have the following constraints between the attitudes,

$$\begin{aligned} S_i &= B_1 R(x_i, \theta_i), \quad i = 1, 2, 3, \\ D_i &= B_2 R(y_i, \phi_i), \quad i = 1, 2, 3. \end{aligned}$$

where  $R(x_i, \theta_i)$  is the rotation about the  $x_i$  axis by the angle  $\theta_i$ , e.g.

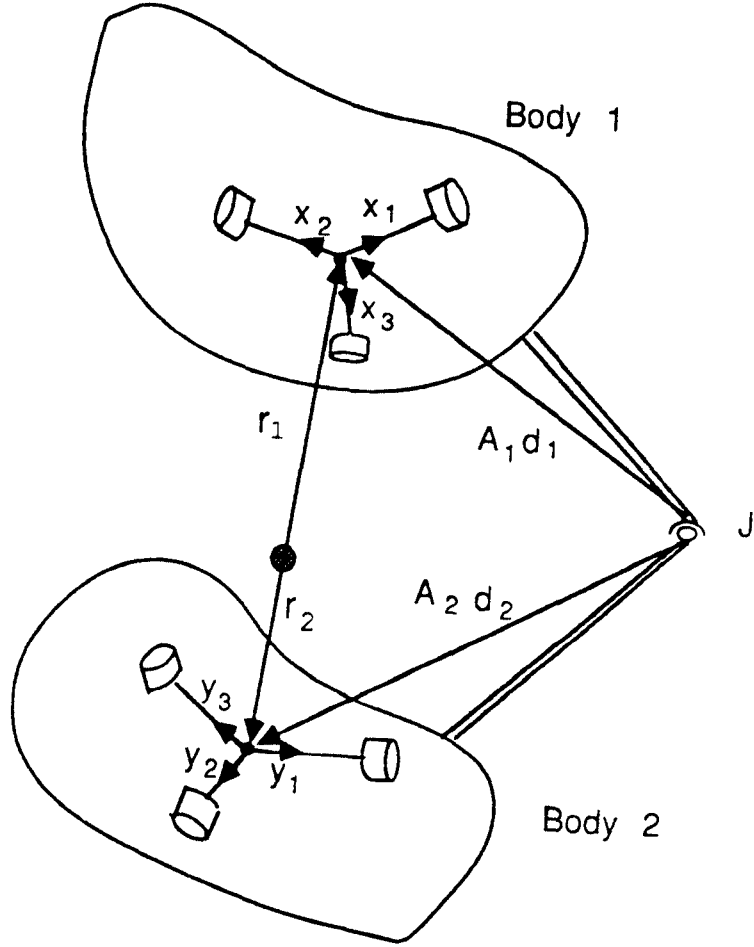


Figure 1. Two Rigid Bodies with Rotors

$$R(x_3, \theta_3) = \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With these constraints, the configuration space is then

$$Q = SO(3) \times (S^1)^3 \times SO(3) \times (S^1)^3.$$

Assume, for convenience, the centers of mass of rotors  $\mathcal{S}_i$  are at the center of mass of carrier body  $B_1$ , and the centers of mass of rotors  $\mathcal{D}_i$  are at the center of mass of the carrier body  $B_2$ . It can be easily found from Fig. 1 that we have the following kinematic constraints,

$$r_{s_i} = r_1, \quad i = 1, 2, 3,$$

$$r_{D_i} = r_2, \quad i = 1, 2, 3,$$

$$r_2 = r_1 - B_1 d_1 + B_2 d_2$$

$$(m_1 + m_s) r_1 = -(m_2 + m_D) r_2$$

with

$$m_s = m_{s_1} + m_{s_2} + m_{s_3},$$

$$m_D = m_{D_1} + m_{D_2} + m_{D_3},$$

where  $m_1, m_{s_i}, m_2, m_{D_i}$  are the masses of the corresponding bodies. By using standard techniques, the total kinetic energy of the system can be written as

$$\begin{aligned} K = & \frac{1}{2} m_1 \|\dot{r}_1\|^2 + \frac{1}{2} \text{tr}(\dot{B}_1 I_1 \dot{B}_1^T) + \frac{1}{2} m_2 \|\dot{r}_2\|^2 + \frac{1}{2} \text{tr}(\dot{B}_2 I_2 \dot{B}_2^T) \\ & + \sum_{i=1}^3 \left( \frac{1}{2} m_{s_i} \|\dot{r}_{s_i}\|^2 + \frac{1}{2} \text{tr}(\dot{S}_i I_{S_i} \dot{S}_i^T) \right) \\ & + \sum_{i=1}^3 \left( \frac{1}{2} m_{D_i} \|\dot{r}_{D_i}\|^2 + \frac{1}{2} \text{tr}(\dot{D}_i I_{D_i} \dot{D}_i^T) \right), \end{aligned}$$

where  $I_1, I_2, I_{S_i}, I_{D_i}$  are the coefficients of inertia of the corresponding bodies. Assuming there is no potential energy, the Lagrangian on the tangent bundle to the unconstrained configuration space  $Q_u$  can be written as

$$\begin{aligned} \tilde{L} = & \frac{1}{2} \text{tr}(\dot{B}_1 I_1 \dot{B}_1^T) + \frac{1}{2} \text{tr}(\dot{B}_2 I_2 \dot{B}_2^T) + \frac{1}{2} \epsilon \|\dot{B}_1 d_1 - \dot{B}_2 d_2\|^2 \\ & + \frac{1}{2} \sum_{i=1}^3 \left( \text{tr}(\dot{S}_i I_{S_i} \dot{S}_i^T) + \text{tr}(\dot{D}_i I_{D_i} \dot{D}_i^T) \right), \end{aligned} \quad (2.1)$$

where

$$\epsilon = \frac{(m_1 + m_s)(m_2 + m_D)}{m_1 + m_s + m_2 + m_D}.$$

Now we let

$$\dot{B}_1 = B_1 \hat{\Omega}_1, \quad \dot{B}_2 = B_2 \hat{\Omega}_2,$$



where the map  $\hat{\cdot}$  is the canonical isomorphism from  $\mathbf{R}^3$  to  $so(3)$ , the skew-symmetric matrices. Since  $S_i = B_1 R(x_i, \theta_i)$ , we have

$$\dot{S}_i = \dot{B}_1 R(x_i, \theta_i) + B_1 \dot{R}(x_i, \theta_i) = B_1(\hat{\Omega}_1 + \hat{s}_i)R(x_i, \theta_i).$$

and thus

$$\Omega_{S_i} = R(x_i, \theta_i)^T \Omega_1 + s_i,$$

where

$$s_1 = \begin{pmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{pmatrix}.$$

We can then write

$$tr(\dot{S}_i I_{S_i} \dot{S}_i^T) = tr\left((\hat{\Omega}_1 + \hat{s}_i)R(x_i, \theta_i)I_{S_i}R(x_i, \theta_i)^T(\hat{\Omega}_1 + \hat{s}_i)^T\right).$$

We naturally assume that the rotors have material symmetry about the axis of rotation, i.e.

$$R(x_i, \theta_i) I_{S_i} R(x_i, \theta_i)^T = I_{S_i},$$

and we get

$$tr(\dot{S}_i I_{S_i} \dot{S}_i^T) = tr\left((\hat{\Omega}_1 + \hat{s}_i)I_{S_i}(\hat{\Omega}_1 + \hat{s}_i)^T\right) = \langle \Omega_1 + s_i, I_{S_i}(\Omega_1 + s_i) \rangle.$$

Similar derivations can be applied to the rotors  $\mathcal{D}_i$ . By substituting these formulae in (2.1), the Lagrangian,  $\bar{L} : TQ \rightarrow \mathbf{R}$ , can then be written as

$$\begin{aligned} & \bar{L}(B_1, \theta_i, B_2, \phi_i, \Omega_1, \dot{\theta}_i, \Omega_2, \dot{\phi}_i, i = 1, 2, 3) \\ &= \frac{1}{2} \langle \Omega_1, J_1 \Omega_1 \rangle + \frac{1}{2} \langle \Omega_2, J_2 \Omega_2 \rangle + \epsilon \langle \Omega_1, \hat{d}_1 B_1^T B_2 \hat{d}_2 \Omega_2 \rangle \\ & \quad + \frac{1}{2} \langle \dot{\Theta}, I^S \dot{\Theta} \rangle + \langle \Omega_1, I^S \dot{\Theta} \rangle + \frac{1}{2} \langle \dot{\Phi}, I^D \dot{\Phi} \rangle + \langle \Omega_2, I^D \dot{\Phi} \rangle. \end{aligned} \quad (2.2)$$

with

$$\begin{aligned}
J_1 &= \mathbf{I}_1 + \epsilon \hat{d}_1^T \hat{d}_1 + \sum_{i=1}^3 \mathbf{I}_{S_i} \\
J_2 &= \mathbf{I}_2 + \epsilon \hat{d}_2^T \hat{d}_2 + \sum_{i=1}^3 \mathbf{I}_{D_i} \\
I^S &= \text{diag}((\mathbf{I}_{S_1})_1, (\mathbf{I}_{S_2})_2, (\mathbf{I}_{S_3})_3), \\
I^D &= \text{diag}((\mathbf{I}_{D_1})_1, (\mathbf{I}_{D_2})_2, (\mathbf{I}_{D_3})_3), \\
\Theta &= \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.
\end{aligned}$$

where  $(\mathbf{I}_{S_i})_j$  denotes the  $j$ -th diagonal element in the moment of inertia matrix  $\mathbf{I}_{S_i}$ . The physical meaning of  $J_1$  is that it is the total moment of inertia of  $\mathcal{B}_1$  plus rotors referred to the joint. This Lagrangian will be used later as a basic entity to derive dynamical equations. Our approach to the derivation of the dynamics is novel. We illustrate this by a simplified example in the next section.

### 3. Euler-Lagrange Equation for Special Orthogonal Group $SO(3)$

We denote an element in the special orthogonal group  $SO(3)$  by a  $3 \times 3$  matrix  $A$  satisfying the identity  $A^T A = I$  and  $\det(A) = 1$ . The elements in the tangent bundle  $TSO(3)$  are usually expressed as  $(A, A\hat{\Omega})$ . This is a *global representation* for the tangent bundle of  $SO(3)$ . On the other hand, we recall that the classical Euler-Lagrange equations are based on local coordinates. In [11], Vershik and Faddeev discuss an invariant form of the Lagrange-D'Alembert Principle which gives the classical Euler-Lagrange equation with external forces in local coordinates. Since our representation for  $TSO(3)$  is global, it is useful to write the form of Lagrange-D'Alembert Principle in that representation. We now show how to derive the dynamical equations for motions on  $SO(3)$ .

Lagrangian Mechanics is about second order equations. Thus we need to find a

representation of the *second tangent bundle* on  $SO(3)$ . We first make the following observation.

$$\frac{d}{dt}A\hat{\Omega} = \dot{A}\hat{\Omega} + A\dot{\hat{\Omega}} = A(\hat{\Omega}\hat{\Omega} + \dot{\hat{\Omega}}).$$

Thus a special second tangent vector must be of the form

$$(A, A\hat{\Omega}, A\hat{\Omega}, A(\hat{\Omega}\hat{\Omega} + \dot{\hat{\Omega}})).$$

In general, let  $(A\hat{u}, W) \in T_{(A, A\hat{\Omega})}TSO(3)$ . In order to have this vector generate a curve

$$(Ae^{t\hat{u}}, Ae^{t\hat{u}}(\hat{\Omega} + t\hat{w})) \in TSO(3),$$

which passes through  $(A, A\hat{\Omega})$  when  $t = 0$ , we must have

$$W = \left. \frac{d}{dt} \right|_{t=0} Ae^{t\hat{u}}(\hat{\Omega} + t\hat{w}) = A(\hat{u}\hat{\Omega} + \hat{w}).$$

Any element in  $TTSO(3)$  can therefore be written as

$$(A, A\hat{\Omega}, A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})). \quad (3.1)$$

Next we look for a canonical representation for an element in  $T^*TSO(3)$ . First, we note that the *trace* pairing

$$\langle A, B \rangle = \frac{1}{2}tr(A^T B),$$

provides us a standard way to define the elements in  $T^*SO(3)$ , i.e.  $A\hat{a} \in T_A^*SO(3)$ , and

$$\langle A\hat{a}, A\hat{u} \rangle = \frac{1}{2}tr(\hat{a}^T A^T A\hat{u}) = \langle a, u \rangle,$$

where  $\langle a, u \rangle$  denotes the Euclidean inner product. Letting  $\omega(A, \dot{A}) = (\alpha, \beta) \in T_{(A, \dot{A})}^*TSO(3)$ ,

$$\omega(A, \dot{A})(A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w})) = \frac{1}{2}tr(\alpha^T A\hat{u}) + \frac{1}{2}tr(\beta^T A(\hat{u}\hat{\Omega} + \hat{w})).$$

In order to have  $a, b \in \mathbf{R}^3$ , such that

$$\omega(A, \dot{A}) \left( A\hat{u}, A(\hat{u}\hat{\Omega} + \hat{w}) \right) = \langle a, u \rangle + \langle b, w \rangle,$$

we ask

$$\alpha = A(\hat{b}\hat{\Omega} + \hat{a})$$

$$\beta = A\hat{b}.$$

Thus we obtain the representation for any element in  $T^*TSO(3)$ ,

$$(A, A\hat{\Omega}, A(\hat{b}\hat{\Omega} + \hat{a}), A\hat{b}). \quad (3.2)$$

These global representations (3.1)-(3.2) of the second tangent bundle and the dual of the second tangent bundle on  $SO(3)$  prove to be useful in finding the derivatives or variations of a function (Lagrangian) on  $TSO(3)$  and in deriving the reduced Poisson bracket. In the following, we state the Lagrange-D'Alembert Principle in terms of these representations. Following an argument similar to the one used in deriving the formula in local coordinates (see [1], pp. 215), we get,

### THEOREM 3.1

On  $TSO(3)$ , let a system be described by a Lagrangian  $L$ . Lagrange-d'Alembert Principle in the invariant form [11] applied to the motions on  $SO(3)$  gives us the Euler-Lagrange equation,

$$\begin{aligned} \langle \frac{d}{dt} D_2 L(A, A\hat{\Omega}), A\hat{u} \rangle &= \langle D_1 L(A, A\hat{\Omega}), A\hat{u} \rangle + \langle \alpha, A\hat{u} \rangle, \\ \forall A\hat{u} &\in T_A SO(3). \end{aligned} \quad (3.3)$$

where  $\alpha$  is the external force. (Here  $D_1, D_2$  are the usual partial Frechét differentials.) ■

The key observation here is that with these representations, the space of vertical tangent vectors is isomorphic to the fibers in  $TSO(3)$ . We omit the details of the above derivation. The only difference between (3.3) and the classical Euler-Lagrange equation is in the *explicit interpretations* of the duality pairing. In the next section, we will use

the procedure of this section as a model for working out the dynamics. We remark that, under this framework, we may mix the local coordinate on some manifold and the parametrizations for  $SO(3)$  in deriving the equations of motion. This is very helpful when we are dealing with systems which have a mixed configuration space (e.g. Cartesian products of a Lie group and a smooth manifold).

#### 4. Dynamical Equations

With the Lagrangian formula (2.2), we now apply Theorem 3.1 to derive the dynamical equations for the multibody dual-spin problem. Let  $L$  be the Lagrangian function expressed in terms of the variables

$$(B_1, \Theta, B_2, \Phi, \dot{B}_1, \dot{\Theta}, \dot{B}_2, \dot{\Phi}) \in TQ.$$

First, we need to find the differential of  $L$  in a form analogous to (3.2). It can be found by the following procedure. Let  $(U_1, U_2, U_3, U_4, W_1, W_2, W_3, W_4) \in T_{(B_1, \Theta, B_2, \Phi, \dot{B}_1, \dot{\Theta}, \dot{B}_2, \dot{\Phi})}TQ$ , which can be written as the form, cf. (3.1),

$$(B_1 \hat{u}_1, u_2, B_2 \hat{u}_3, u_4, B_1(\hat{u}_1 \hat{\Omega}_1 + \hat{w}_1), w_2, B_2(\hat{u}_2 \hat{\Omega}_2 + \hat{w}_3), w_4).$$

It generates a curve in  $TQ$  given by

$$(B_1 e^{t\hat{u}_1}, \Theta + tu_2, B_2 e^{t\hat{u}_2}, \Phi + tu_4, B_1 e^{t\hat{u}_1}(\hat{\Omega}_1 + t\hat{w}_1), \dot{\Theta} + tw_2, B_2 e^{t\hat{u}_2}(\hat{\Omega}_2 + t\hat{w}_3), \dot{\Phi} + tw_4).$$

Here we have

$$\begin{aligned}
& dL \cdot (U_1, U_2, U_3, U_4, W_1, W_2, W_3, W_4) \\
&= \frac{d}{dt} \Big|_{t=0} L(B_1 e^{t\hat{u}_1}, \Theta + tu_2, B_2 e^{t\hat{u}_2}, \Phi + tu_4, \\
&\quad B_1 e^{t\hat{u}_1}(\hat{\Omega}_1 + t\hat{w}_1), \dot{\Theta} + tw_2, B_2 e^{t\hat{u}_2}(\hat{\Omega}_2 + t\hat{w}_3), \dot{\Phi} + tw_4) \\
&= \frac{d}{dt} \Big|_{t=0} \bar{L}(B_1 e^{t\hat{u}_1}, \Theta + tu_2, B_2 e^{t\hat{u}_2}, \Phi + tu_4, \\
&\quad \Omega_1 + tw_1, \dot{\Theta} + tw_2, \Omega_2 + tw_3, \dot{\Phi} + tw_4) \\
&= \langle D_{B_1} \bar{L}, B_1 \hat{u}_1 \rangle + \langle \frac{\partial \bar{L}}{\partial \Theta}, u_2 \rangle + \langle D_{B_2} \bar{L}, B_2 \hat{u}_3 \rangle + \langle \frac{\partial \bar{L}}{\partial \Phi}, u_4 \rangle \\
&\quad + \langle \frac{\partial \bar{L}}{\partial \Omega_1}, w_1 \rangle + \langle \frac{\partial \bar{L}}{\partial \dot{\Theta}}, w_2 \rangle + \langle \frac{\partial \bar{L}}{\partial \Omega_2}, w_3 \rangle + \langle \frac{\partial \bar{L}}{\partial \dot{\Phi}}, w_4 \rangle.
\end{aligned}$$

The canonical form for  $dL(B_1, \Theta, B_2, \Phi, \dot{B}_1, \dot{\Theta}, \dot{B}_2, \dot{\Phi})$  is, cf. (3.2),

$$(B_1(\hat{b}_1 \hat{\Omega}_1 + \hat{a}_1), a_2, B_2(\hat{b}_3 \hat{\Omega}_2 + \hat{a}_3), a_4, B_1 \hat{b}_1, b_2, B_2 \hat{b}_3, b_4).$$

Let  $N_1, N_2$  be given by the formula

$$D_{B_1} \bar{L} = B_1 \hat{N}_1, \quad D_{B_2} \bar{L} = B_2 \hat{N}_3,$$

we have

$$\begin{aligned}
a_1 &= N_1, \quad a_2 = \frac{\partial \bar{L}}{\partial \Theta}, \quad a_3 = N_3, \quad a_4 = \frac{\partial \bar{L}}{\partial \Phi}, \\
b_1 &= \frac{\partial \bar{L}}{\partial \Omega_1}, \quad b_2 = \frac{\partial \bar{L}}{\partial \dot{\Theta}}, \quad b_3 = \frac{\partial \bar{L}}{\partial \Omega_2}, \quad b_4 = \frac{\partial \bar{L}}{\partial \dot{\Phi}},
\end{aligned}$$

and we get the form of elements in (3.3),

$$\begin{aligned}
D_1 L &= \left( B_1 \left( \widehat{\frac{\partial \bar{L}}{\partial \Omega_1}} \hat{\Omega}_1 + \hat{N}_1 \right), \frac{\partial \bar{L}}{\partial \Theta}, B_2 \left( \widehat{\frac{\partial \bar{L}}{\partial \Omega_2}} \hat{\Omega}_2 + \hat{N}_3 \right), \frac{\partial \bar{L}}{\partial \Phi} \right), \\
D_2 L &= \left( B_1 \widehat{\frac{\partial \bar{L}}{\partial \Omega_1}}, \frac{\partial \bar{L}}{\partial \dot{\Theta}}, B_2 \widehat{\frac{\partial \bar{L}}{\partial \Omega_2}}, \frac{\partial \bar{L}}{\partial \dot{\Phi}} \right).
\end{aligned} \tag{4.1}$$

Next we model the external force  $\alpha$ . Let  $T^J$  be the torque on the ball-in-socket joint in the body 2 frame,  $T_{S_i}, T_{D_i}$  denote the torques exerted on the driven rotors and damping

rotors respectively. Letting  $B = B_1^T B_2$ , the *shape* of the system, we can write  $\alpha$  in the form

$$\alpha = \left( B_1(-\widehat{BT}^J), T^S, B_2\widehat{T}^J, T^D \right).$$

where

$$T^S = \begin{pmatrix} (T_{S_1})_1 \\ (T_{S_2})_2 \\ (T_{S_3})_3 \end{pmatrix}, \quad T^D = \begin{pmatrix} (T_{D_1})_1 \\ (T_{D_2})_2 \\ (T_{D_3})_3 \end{pmatrix}.$$

with the notation  $(T_{S_i})_j$  denoting the  $j$ -th component of the vector  $T_{S_i}$ .

We are ready to apply formula (3.3). The first component is worked out here. The others can be found in a similar way. From

$$B_1 \hat{\Omega}_1 \frac{\partial \widehat{L}}{\partial \Omega_1} + B_1 \frac{d}{dt} \frac{\partial \widehat{L}}{\partial \Omega_1} - B_1 \left( \frac{\partial \widehat{L}}{\partial \Omega_1} \hat{\Omega}_1 + \hat{N}_1 + (-\widehat{BT}^J) \right) = 0,$$

we get

$$\frac{d}{dt} M_1 = -\Omega_1 \times M_1 + N_1 - BT^J,$$

where  $M_1 = \frac{\partial \widehat{L}}{\partial \Omega_1}$ . This can be rewritten in terms of  $\Omega_1$ , etc. Explicitly, we get the dynamical equations of the system in terms of variables in  $TQ$ .

$$J_1 \dot{\Omega}_1 + I^S \ddot{\Theta} + \epsilon \hat{d}_1 B \hat{d}_2 \dot{\Omega}_2 = -\Omega_1 \times (J_1 \Omega_1 + I^S \dot{\Theta}) - \epsilon \hat{d}_1 B \hat{\Omega}_2 \hat{d}_2 \Omega_2 - BT^J,$$

$$J_2 \dot{\Omega}_2 + I^D \ddot{\Phi} + \epsilon \hat{d}_2 B^T \hat{d}_1 \dot{\Omega}_1 = -\Omega_2 \times (J_2 \Omega_2 + I^D \dot{\Phi}) - \epsilon \hat{d}_2 B^T \hat{\Omega}_1 \hat{d}_1 \Omega_1 + T^J,$$

$$I^S (\dot{\Omega}_1 + \ddot{\Theta}) = T^S,$$

$$I^D (\dot{\Omega}_2 + \ddot{\Phi}) = T^D,$$

$$\dot{B}_1 = B_1 \hat{\Omega}_1,$$

$$\dot{B}_2 = B_2 \hat{\Omega}_2.$$

(4.2a)

For the realization of the multibody dual-spin control structure, the following *feedback laws* are used,

$$\begin{aligned} T^S &= I^S \dot{\Omega}_1, \\ T^D &= -\beta \dot{\Phi}, \\ T^J &= -\gamma(\Omega_2 - B^T \Omega_1), \end{aligned} \tag{4.2b}$$

where  $\beta$ , and  $\gamma$  are positive definite matrices. The first equation in (4.2b) makes the relative angular velocities between the driven rotors and  $\mathcal{B}_1$  be constants. The other two equations signify damping torques on the damping rotors and the joint. With these feedback laws and the following transformation of coordinates to conjugate momenta,

$$p_1 = J_1 \Omega_1 + \epsilon \hat{d}_1 B \hat{d}_2 \Omega_2, \tag{4.3a}$$

$$p_2 = I^S \dot{\Theta}, \tag{4.3b}$$

$$p_3 = (J_2 - I^D) \Omega_2 + \epsilon \hat{d}_2 B^T \hat{d}_1 \Omega_1, \tag{4.3c}$$

$$p_4 = I^D(\Omega_2 + \dot{\Phi}), \tag{4.3d}$$

we can express the dynamical equations (4.2) in terms of  $p_i$  variables,

$$\begin{aligned} \dot{p}_1 &= -\tilde{\Omega}_1 \times (p_1 + p_2) - \epsilon \widehat{\hat{d}_1 \tilde{\Omega}_1} B \hat{d}_2 \tilde{\Omega}_2 - \gamma(\tilde{\Omega}_1 - B \tilde{\Omega}_2), \\ \dot{p}_2 &= 0. \\ \dot{p}_3 &= -\tilde{\Omega}_2 \times (p_3 + p_4) - \epsilon \widehat{\hat{d}_2 \tilde{\Omega}_2} B^T \hat{d}_1 \tilde{\Omega}_1 - \gamma(\tilde{\Omega}_2 - B^T \tilde{\Omega}_1) + \beta \tilde{\Phi}, \\ \dot{p}_4 &= -\beta \tilde{\Phi}, \\ \dot{B} &= B \hat{\Omega}_2 - \hat{\Omega}_1 B. \end{aligned} \tag{4.4}$$

Here  $\tilde{\Omega}_1$ ,  $\tilde{\Omega}_2$ , and  $\tilde{\Phi}$  are the expressions of  $\Omega_1$ ,  $\Omega_2$ , and  $\dot{\Phi}$  in terms of  $p_i$ , respectively. These expressions can be found through (4.3).



## 5. Asymptotic Stability

Equations (4.4) describe the dynamical behavior of the system under investigation. We are now ready to establish an asymptotic stability theorem. First, in terms of the conjugate momentum variables (4.3), the Lagrangian can be written as

$$\begin{aligned}\bar{L} = & \frac{1}{2} \langle \tilde{\Omega}_1, p_1 \rangle + \frac{1}{2} \langle (I^S)^{-1} p_2, p_2 \rangle + \frac{1}{2} \langle \tilde{\Omega}_2, p_3 \rangle \\ & + \frac{1}{2} \langle (I^D)^{-1} p_4, p_4 \rangle + \langle \tilde{\Omega}_1, p_2 \rangle.\end{aligned}$$

We define the function

$$V = \bar{L} - \langle \tilde{\Omega}_1, p_2 \rangle. \quad (5.1)$$

It can be shown that the directional derivative of  $V$  along a trajectory is

$$\frac{dV}{dt} = -\langle \tilde{\Phi}, \beta \tilde{\Phi} \rangle - \langle \tilde{\Omega}_2 - B^T \tilde{\Omega}_1, \gamma (\tilde{\Omega}_2 - B^T \tilde{\Omega}_1) \rangle.$$

Thus this function is a Lyapunov function candidate. This is an analog of the *core energy* used in [7] to justify the energy-sink method and used in [8] to prove asymptotic stability. In the absence of damping ( i.e.  $\gamma = \beta = 0$  ), the system is a hamiltonian system with hamiltonian function  $V$ . The hamiltonian structure has been worked out in the ongoing Ph.D. dissertation of L.-S. Wang [12]. It is a generalization of the one derived by Grossman, Krishnaprasad, and Marsden in [6]. The hamiltonian structure is noncanonical.

In the following, we shall use an argument similar to the one in [8] to do the stability analysis. It can be shown that the function

$$\|p_1 + p_2 + B(p_3 + p_4)\|^2$$

is invariant under the motion. It is the Casimir invariant of the hamiltonian limit. Since  $p_2$  is a constant along any trajectory (as achieved by the feedback law (4.2b)), we define the *momentum variety*

$$M_{p_2}^\mu = \{(p_1, p_3, p_4, B) \in \mathbf{R}^9 \times SO(3) : \|p_1 + p_2 + B(p_3 + p_4)\|^2 = \mu^2\}, \quad (5.3)$$

which is parametrized by  $p_2$ , and  $\mu$ . The dynamical motion leaves the momentum variety invariant. This will be the domain of our analysis later on. Let

$$\begin{aligned}\mathcal{R} &= \{ (p_1, p_3, p_4, B) \in M_{p_2}^\mu : \frac{dV}{dt} = 0 \}, \\ &= \{ (p_1, p_3, p_4, B) \in M_{p_2}^\mu : (I^D)^{-1} p_4 - \tilde{\Omega}_2 = 0, \tilde{\Omega}_2 - B^T \tilde{\Omega}_1 = 0 \}.\end{aligned}\tag{5.4}$$

It can be shown that  $\mathcal{R}$  is parametrized by  $(p_4, p, B)$  as

$$\begin{aligned}p_1 &= p - (BJ_2 + \epsilon \hat{d}_1 B \hat{d}_2)(I^D)^{-1} p_4, \\ p_3 &= \epsilon \hat{d}_2 B^T \hat{d}_1 J_1^{-1} p + (J_2 - I^D - \epsilon \hat{d}_2 B^T \hat{d}_1 J_1^{-1} B J_2)(I^D)^{-1} p_4,\end{aligned}\tag{5.5}$$

where  $p$  satisfies  $\|p + p_2\|^2 = \mu^2$ . From (4.4), the set of equilibria in  $M_{p_2}^\mu$  can be written as

$$\begin{aligned}\sum_{\mu, p_2} &= \left\{ (p_1, p_3, p_4, B) \in \mathbb{R}^9 \times SO(3) : \|p + p_2\|^2 = \mu^2, \right. \\ &\quad p_1 = p - (BJ_2 + \epsilon \hat{d}_1 B \hat{d}_2)(I^D)^{-1} p_4, \\ &\quad p_3 = \epsilon \hat{d}_2 B^T \hat{d}_1 J_1^{-1} p + (J_2 - I^D - \epsilon \hat{d}_2 B^T \hat{d}_1 J_1^{-1} B J_2)(I^D)^{-1} p_4, \\ &\quad (I^D)^{-1} p_4 - \tilde{\Omega}_2 = 0, \quad \tilde{\Omega}_2 - B^T \tilde{\Omega}_1 = 0, \\ &\quad -\tilde{\Omega}_1 \times (p_1 + p_2) - \epsilon \widehat{\hat{d}_1 \tilde{\Omega}_1} B \hat{d}_2 \tilde{\Omega}_2 = 0, \\ &\quad \left. -\tilde{\Omega}_2 \times (p_3 + p_4) - \epsilon \widehat{\hat{d}_2 \tilde{\Omega}_2} B^T \hat{d}_1 \tilde{\Omega}_1 = 0 \right\}\end{aligned}$$

Now we characterize the maximal invariant set in  $\mathcal{R}$ . We ask that the directional derivative of the functions  $(I^D)^{-1} p_4 - \tilde{\Omega}_2$  and  $\tilde{\Omega}_2 - B^T \tilde{\Omega}_1$  along the trajectory be 0. It can be shown that this condition is equivalent to

$$\dot{\tilde{\Omega}}_1 = 0, \quad \dot{\tilde{\Omega}}_2 = 0.$$

From (4.3), we have

$$\begin{pmatrix} p_1 \\ p_3 \end{pmatrix} = \begin{pmatrix} J_1 & \epsilon \hat{d}_1 B \hat{d}_2 \\ \epsilon \hat{d}_2 B^T \hat{d}_1 & J_2 - I^D \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = J \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}.$$

After further manipulations, we obtain the formula,

$$\frac{d}{dt} \begin{pmatrix} \tilde{\Omega}_1 \\ \tilde{\Omega}_2 \end{pmatrix} = J^{-1} \begin{pmatrix} -\tilde{\Omega}_1 \times (p_1 + p_2) - \epsilon \widehat{\hat{d}_1} \tilde{\Omega}_1 B \hat{d}_2 \tilde{\Omega}_2 - \epsilon \hat{d}_1 \dot{B} \hat{d}_2 \tilde{\Omega}_2 \\ -\tilde{\Omega}_2 \times (p_3 + p_4) - \epsilon \widehat{\hat{d}_2} \tilde{\Omega}_2 B^T \hat{d}_1 \tilde{\Omega}_1 - \epsilon \hat{d}_2 \dot{B}^T \hat{d}_1 \tilde{\Omega}_1 + \beta \dot{\Phi} \end{pmatrix}.$$

On  $\mathcal{R}$ ,  $\dot{B} = 0$ ,  $\dot{\Phi} = 0$ , we thus proved that the maximal invariant set is exactly the same as the set  $\sum_{\mu, p_2}$  of equilibria for (4.4). From LaSalle's theorem [9], we know that a trajectory of motion will approach the maximal invariant set in the limit. We can thus conclude the following theorem.

### THEOREM 5.1

The mechanical system (4.4) asymptotically approaches one of the stable equilibria in  $\sum_{\mu, p_2}$ , or the equilibria of the limiting hamiltonian system.

■

The limiting motions of the system, i.e. the equilibria, are exactly the same notion as the *relative equilibria* discussed in [13]. The techniques for finding the relative equilibria there can be applied to our problem by additionally including one *gyroscopic* term in the augmented potential function.

## 6. Conclusions

We established a version of the Euler-Lagrange equation for the motions on  $SO(3)$  from the invariant form of the Lagrange-D'Alembert Principle. This scheme can be used to derive dynamical equations for many-rigid-body problems. It can also be applied to certain continuum mechanical problems, e.g. the special Cosserat rod. We applied the method to a multibody analog of the classical dual-spin problem and obtained the dynamical equations. It is then proved that with a damping mechanism on the ball-in-socket joint and the damping rotors, we have asymptotic stability. The driven rotors are expected to perform the role of attitude acquisition. This problem will be studied further in the future.

## REFERENCES

- [1] ABRAHAM, R. & J.E. MARSDEN, *Foundations of Mechanics*, 2nd ed., Revised, Enlarged, Reset , Benjamin/Cummings, Reading, 1978.
- [2] ALVAREZ, Gloria Sanchez de, "Geometric Methods of Classical Mechanics Applied to Control Theory," University of California, Berkeley, Ph.D. Thesis, 1986.
- [3] BAILLIEUL, J., "Controllability and Observability of Polynomial Dynamical Systems," *Nonlinear Analysis, Theory, Methods, and Applications*, 5, 5 (1981), pp. 543-552.
- [4] BONNARD, B., "Controllability of Mechanical Control Systems on Lie Groups," Laboratoire d'Automatique de Grenoble, Ecole Nationale Supérieure d'Ingénieurs, Note 82-08, France, Jan. 1982.
- [5] CROUCH, P.E., "Spacecraft Attitude Control and Stabilization: Applications of Geometric Control Theory to Rigid Body Models," *IEEE Trans. Automatic Control*, 29 (1984), pp. 321-331.
- [6] GROSSMAN, R., P.S. KRISHNAPRASAD & J.E. MARSDEN, "The Dynamics of Two Coupled Three Dimensional Rigid Bodies," in *Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits*, F. SALAM & M. LEVI, eds., SIAM PUBL., Philadelphia, 1988, pp. 373-378 .
- [7] HUBERT, C., "Dynamics of the Generalized Dual-Spin Turn," *RCA Rev.*, 41 (1980), pp. 449-471.
- [8] KRISHNAPRASAD, P.S., "Lie-Poisson Structures, Dual-spin Spacecraft and Asymptotic Stability," *Nonlinear Analysis: Theory, Methods, and Applications*, 7 (1984), pp. 1011-1035.
- [9] LASALLE, J.P., *The Stability of Dynamical Systems*, SIAM, 1976.

- [10] PATRICK, G., "The Dynamics of Two Coupled Rigid Bodies in Three Space," in *Dynamics and Control of Multibody Systems*, J.E. MARSDEN, P.S. KRISHNAPRASAD & J.C. SIMO, eds., AMS, Providence, 1989.
- [11] VERSHIK, A.M. & L.D. FADDEEV, "Lagrangian Mechanics in Invariant Form," *Sel. Math. Sov.*, 1, 4 (1981), pp. 339–350.
- [12] WANG, L.-S., "Dynamics and Control of Systems of Coupled Bodies," Ph.D. Dissertation, 1990, In preparation.
- [13] WANG, L.-S. & P.S. KRISHNAPRASAD, "Relative Equilibria of Two Rigid Bodies connected by a Ball-in-Socket Joint," *Proc. of the 1989 IEEE Conference on Decision and Control*, Tampa, FL (Dec. 1989), pp. 692–697.